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Computational Methods for Efficient Structural Reliability and Reliability Sensitivity Analysis

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This paper proposes an efficient, adaptive importance sampling (AIS) method that can be used to compute component and system reliability and reliability sensitivities. The AIS approach uses a sampling density that is proportional to the joint probability density function of the random variables. Starting from an initial approximate failure domain, sampling proceeds adaptively and incrementally to reach a sampling domain that is slightly greater than the failure domain to minimize oversampling in the safe region. Several reliability sensitivity coefficients are proposed that can be computed directly and easily from the previous AIS-based failure points. These sensitivities can be used to identify key random variables and to adjust a design to achieve reliability-based objectives. The proposed methodology is demonstrated using a turbine blade reliability analysis problem.

Nomenclature

$E[.]$ = expected value
F = cumulative distribution function (CDF)
f = probability density function
g = limit-state function
N = number of simulation samples
N^+ = number of failures in a sampling
n = number of random variables
$P[.]$ = probability of
p, p_f = probability of failure
S = importance sampling region
S_μ = mean sensitivity coefficient
S_σ = sigma (standard deviation) sensitivity coefficient
u, v = standard normal variable
X = random variable
x = realization of X
Z = performance function; response function
z = realization of Z
α_i = direction cosines to the most probable point (MPP)
β = minimum distance, safety index
γ = error bound
Δ = indicates an incremental amount
θ = statistical distribution parameter (μ, σ)
κ_i = curvatures at the MPP
μ = mean value
σ = standard deviation
Φ = standard normal CDF
Ω = failure region

Superscript

$*$ = most probable point

Introduction

THE analysis and design of advanced, high-performance structures requires accurate, computer-intensive calculations using numerical algorithms such as finite element methods. To ensure high reliability, numerical simulation and reliability testing of structures are needed. This paper deals with the numerical simulation issues. A straightforward Monte Carlo method, although very useful for reliability analysis of simple problems, becomes impractical for large numerical models because it requires a large number of random simulations. Thus, in structural reliability and risk assessment, one of the difficulties and challenges is the development of efficient and accurate reliability and reliability sensitivity calculation algorithms for analyzing complicated models.

In this paper, the efficient methods developed for a structural risk assessment software system, Numerical Evaluation of Stochastic Structures Under Stress (NESSUS), are presented, with a major portion devoted to a reliability sensitivity analysis methodology based on a proposed adaptive importance sampling (AIS) method. NESSUS, which integrates reliability methods with finite element methods, is a probabilistic analysis tool to simulate stochastic structures operating under severe loading conditions such as space propulsion system structures.

Efficient and accurate computational reliability methods for a single limit-state (or performance function) have been developed and successfully applied to aerospace propulsion structures.[1–3] Computational methods have also been developed for system reliability problems.[4] Structural reliability sensitivity calculation methods are well developed for single limit-state problems.[5] For multiple limit states, available reliability sensitivity calculation methods are primarily based on probability bounds or asymptotic results (i.e., for $\beta \to \infty$) of series and parallel systems.[5] In addition, reliability sensitivity measures are typically based on certain approximation points, e.g., design points, and are valid only in the first-order approximation sense. In this paper, a method is proposed to directly and accurately calculate the component and the system reliability sensitivities based on the AIS-generated sampling points.

Review of Limit-State Reliability Analysis Methods

In structural reliability analysis, a limit-state or failure function $g(X)$ is defined. The g function is a function of a vector of basic random variables, $X = (X_1, X_2, \ldots, X_n)$, with $g(X) = 0$ being the limit-state surface that separates the variable space into two regions: failure ($g \le 0$) and safe ($g > 0$). Given the joint probability density function (PDF) $f_x(x)$, the probability of failure is the probability in the failure domain Ω and is given by

$$p_f = \int_\Omega \cdots \int f_x(x) \, \mathrm{d}x \tag{1}$$

The preceding multiple integral can be computed using a straightforward standard Monte Carlo procedure. However, when the g function is complicated, requiring intensive finite element analysis, and p_f is small, the random sampling procedure becomes impractical for analysis and design.

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A widely used procedure described later performs reliability and reliability sensitivity analyses in a coordinate system of an independent, standardized normal vector $\boldsymbol{u}$.[6] By transforming $g(\boldsymbol{X})$ to $g(\boldsymbol{u})$ using a proper distribution transformation, the most probable point (MPP) in the u space $\boldsymbol{u}^*$ is a point that defines the minimum distance β from the origin ($\boldsymbol{u} = 0$) to the limit-state surface $g(\boldsymbol{u}) = 0$. This point, also called design point in the structural reliability literature, is most probable (in the u space, not necessarily in the X space) in the sense that it has maximum joint probability density function (PDF) on $g(\boldsymbol{u}) = 0$.

Once the MPP is identified and an approximate linear or quadratic function is developed around the MPP, the probability estimate can be calculated easily. The sensitivity factors α_i are the directional cosines of the vector from the origin to the MPP. The sensitivity factors provide first-order information on the importance of the individual random variable. Other sensitivity measures with respect to a distribution parameter (mean or standard deviation) or a limit-state function parameter can be estimated based on the sensitivity factors and the distribution transformation.[5] Thus, for complicated g functions, the primary computational effort is typically spent on locating the MPP and developing the polynomial function.

Advanced Mean-Value-Based Method

Based on the limit-state concept, many reliability analysis algorithms have been developed.[1,5–7,17] This section summarizes the advanced mean-value (AMV) based method that has been demonstrated to be well suited for dealing with complicated, implicitly defined g functions.[1]

Given a response or performance function $Z(\boldsymbol{X})$, one of the goals in probabilistic analysis is to compute its cumulative distribution function (CDF). For a reliability analysis problem, a limit-state function is defined for each selected response level z, i.e.,

$$g(\boldsymbol{X}) = Z(\boldsymbol{X}) - z = 0 \tag{2}$$

such that $P[g \le 0] = P[Z \le z]$.

The AMV method provides an approximate CDF analysis capability with slightly more calculations than a mean-value, second moment analysis. In addition, the AMV-based CDF provides information on the nonlinearity of the g function to detect potential numerical problems.[1,7]

The AMV analysis starts with a mean-value (MV) model defined in Eq. (3) and leads to an accurate CDF analysis:

$$Z_{\text{MV}} = Z(\mu_x) + \sum_{i=1}^{n} \left(\frac{\partial Z}{\partial X_i}\right)\bigg|_{\mu_x} (X_i - \mu_{x_i}) = a_0 + \sum_{i=1}^{n} a_i X_i \tag{3}$$

The derivatives in Eq. (3) can be obtained by numerical differentiation or other sensitivity calculation methods.

The AMV model is defined as

$$Z_{\text{AMV}} = Z_{\text{MV}} + H(Z_{\text{MV}}) \tag{4}$$

where $H(Z_{\text{MV}})$ is defined as the difference between the values of Z_{MV} and Z calculated at the most probable point locus (MPPL) of Z_{MV}. The MPPL is defined by connecting all the MPPs for different z values. The MPPL of Z_{MV} is generally a nonlinear curve in the u space. For a complicated structural analysis, the construction of Eq. (3) may be time consuming, but its MPPL can be computed easily.

The computational steps for computing the CDF value are as follows: 1) Based on Z_{MV}, compute the MPP, $\boldsymbol{x}^*$, for a selected CDF value. 2) Compute $Z(\boldsymbol{x}^*)$ to update z for the selected CDF value. Given the MV model, the required number of Z function calculations equals the number of selected CDF values.

The accuracy of the preceding CDF solution depends on the quality of the MPPL from Z_{MV}, i.e., the solution is good if the approximate locus is close to the exact one. In general, the AMV solution can be improved by using an improved expansion point, which can be done typically by an optimization procedure or an iteration procedure. Based initially on Eq. (3) and by keeping track of the MPPL,[8] the exact MPP for a limit state $Z = z$ can be computed to establish the first-order AMV+ model defined as

$$Z_{\text{AMV+}} = Z(\boldsymbol{x}^*) + \sum_{i=1}^{n} \left(\frac{\partial Z}{\partial X_i}\right)\bigg|_{\boldsymbol{x}^*} (X_i - x_i^*) = b_0 + \sum_{i=1}^{n} b_i X_i = z \tag{5}$$

where $\boldsymbol{x}^*$ is the converged MPP for $Z = z$.

It is important to note that the variables X_i are generally non-normal and dependent; therefore the preceding AMV+ model, which is linear in the X space, is different from the first-order reliability method (FORM) model,[5] which is linear in the u space. An advantage of Eq. (5) is that it provides an initial estimate of the curvature (about the MPP) in the u space that can be used both for approximate probability of failure analysis and for the AIS analysis described later. It is particularly useful when the curvature is primarily caused by the non-normal to normal transformation, which has been observed to occur for many engineering application problems.

Multiple Limit States

For system problems that involve multiple limit states, the preceding limit-state reliability analysis methods can be applied to all of the g functions. Accurate system reliability analysis usually requires additional calculation of the joint probabilities. Because of the complexities involved, approximate and simplified methods such as the probability bounding approaches and asymptotic results are commonly used.[5,6] From the literature, the accuracies of these approximation methods seem to be problem dependent and must be used with caution.

System reliability sensitivity analyses also are much more difficult than for a single limit state because there are multiple MPPs. The sensitivity factors derived from the individual g function cannot be used to derive the system reliability sensitivities because the contribution from the individual g function cannot be quantified easily. This paper proposes using the approximated limit states to develop importance sampling regions. Both the system reliability and its sensitivity are computed based on the AIS samples and eliminate the computational difficulties presented in the approximate analytical approach.

Adaptive Importance Sampling

First consider a single limit state. If the true limit-state surface cannot be well represented by a low-order polynomial, the probability estimate may be inaccurate. Therefore, it is useful to improve or check the approximate limit-state surface solution by sampling in the critical regions based on $Z_{\text{AMV+}}$. Several importance sampling schemes have been proposed that focus sampling in the probability-critical regions to increase sampling efficiency.[9–14] An adaptive importance sampling (AIS) concept originally developed for a rotor instability analysis problem has been extended recently for solving general system reliability problems.[4,15,16]

The AIS approach has the following features: 1) the sampling density is proportional to the joint PDF of the random variables, 2) the initial sampling domain is the failure side of the limit-state surface constructed from an approximate, parabolic surface, 3) additional samples are added after incrementally changing the curvatures of the parabolic surface at the most probable point, and 4) the final sampling domain contains, but is only slightly greater than, the failure domain.

The goal of the preceding AIS approach is to minimize oversampling in the safe region. Ideally, one does not want to perform any sampling in the safe region. However, the only way to detect the limit-state surface is by crossing the limit-state surface. To minimize unnecessary sampling, the chosen adaptive surface for AIS should be flexible enough to represent closely the exact limit-state surface. Significant efficiency gain relative to the conventional

Monte Carlo method can be achieved if the sampling is performed in the regions that are very close to the failure regions.

There are several possibilities in choosing and varying the AIS limit-state surface. For example, linear, quadratic, and (hyper-) sphere limit-state surfaces could be used for AIS sampling. A parabolic surface offers good performance in efficiency and robustness and is demonstrated in this paper using a numerical example. The proposed AIS concept is illustrated in Fig. 1, in which the parabolic surface is rotationally symmetric about the vector OP that passes through the origin and the MPP. In general, the sampling domain can be adjusted by increasing or decreasing the curvatures of the parabolic surface. By decreasing the curvatures to the extreme ($-\infty$), the sampling region will cover the entire space. However, if β is correctly calculated, the region inside the sphere need not be sampled. Therefore, the AIS bound is a sphere, as illustrated in Fig. 1. For additional checking, the distance to the MPP can also be reduced in case the calculated β is in error.

AIS for a Single Limit State

Assume that an initial second-order limit state is available:

$$g(u) = \nabla g(u^*)^T (u - u^*) + 1/2 (u - u^*)^T H(u^*) (u - u^*) \quad (6)$$

where $\nabla g(u)$ is the gradient at the MPP, and H is the Hessian matrix containing second-order partial derivatives. Because the preceding equation will be used to define an initial sampling domain for the AIS, only a rough estimate of H is needed. The adopted approach for this paper is to use the linear $g(X)$ function in Eq. (5) and transform it to a second-order $g(u)$ function. Other methods such as the curve-fitting approach may be used.[17]

The transformation procedure to develop a parabolic surface based on Eq. (6) is well documented.[5,18] The parabolic surface can be written as

$$g = \beta - v_n + \sum_{i=1}^{n-1} \lambda_i v_i^2 \quad (7)$$

in which v_i are independent standardized normal variables, and λ_i are related to the main curvatures by the relation $\kappa_i = 2\lambda_i$. The proposed AIS procedure uses Eq. (7) to develop the initial adaptive sampling boundary and generate the samples in the $g \le 0$ domain. The initial sampling region is S. The probability in S, denoted as p_s, can be numerically computed using a second-order reliability method or a convolution method.[4,18]

To change the limit-state surface for additional sampling, the λ_i ($i = 1, n-1$) are changed to λ'_i. The changes in λ_i can be made individually or simultaneously based on a selected probability increment Δp_s.[4] Define the perturbed limit-state surface as

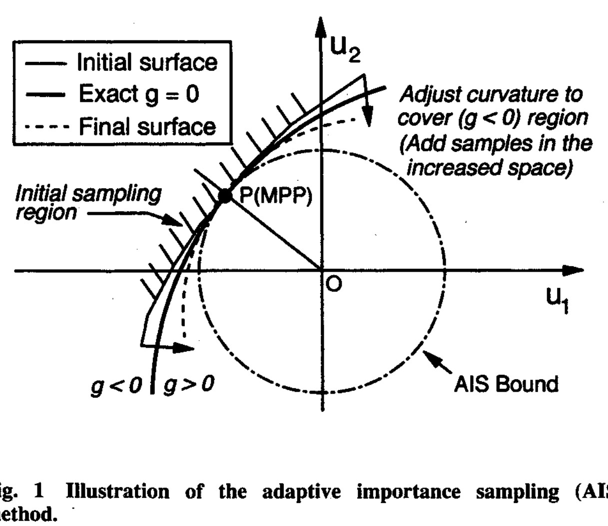

Fig. 1 Illustration of the adaptive importance sampling (AIS) method.

$$g' = \beta - v_n + \sum_{i=1}^{n-1} \lambda'_i v_i^2 \quad (8)$$

where the prime indicates a perturbed condition. The sampling region is now $S' = S + \Delta S$.

Let N_1 be the initial number of samples determined based on a specified confidence interval and an error bound. During the sampling process, this number needs to be adjusted based on the calculated probability of failure in the initial sampling region S. The number of failure points in S is denoted as N_1^+. Given λ'_i, the sampling is applied to an increased sampling ΔS region. A predetermined number of samples ΔN in the ΔS region is calculated using the equation

$$\Delta N = (p'_s - p_s) N_1 = \Delta p_s N_1 \quad (9)$$

where p'_s is the updated probability in the perturbed region S'. By sampling the original g function, the number of failure points ΔN^+ within ΔS can be determined, and the updated probability estimate becomes

$$p_f \approx p'_s \cdot \hat{p} = p'_s \cdot \frac{N_1^+ + \Delta N^+}{N_1 + \Delta N} \quad (10)$$

where $\hat{p}$ is the estimate of the conditional probability of failure in S', i.e., $P[g \le 0 / S']$. The perturbation procedure is repeated until no additional failure points are observed after at least one perturbation and the result satisfies a convergence criteria described later.

Required Number of Samples

Assuming that $\hat{p}$ has a normal distribution, for a $(1-\alpha)$ confidence interval, the error bound γ (in percentage) is

$$\gamma = 100 \cdot \Phi^{-1}\left(1 - \frac{\alpha}{2}\right)\sqrt{\frac{1-\hat{p}}{\hat{p}N}} \quad (11)$$

where $\Phi(\cdot)$ is the standard normal distribution function. Equation (11) can be used to calculate the required samples N given an estimated $\hat{p}$. Ideally, we wish to apply AIS to a region very close to the exact failure region, resulting in $\hat{p} \approx 1$. In practice, however, the goal of $\hat{p} = 1$ cannot be achieved because the sampling region must be greater than the failure region to indicate that the failure region has been covered sufficiently.

The convergence procedure used in this paper is as follows:

1) Generate samples in the initial sampling domain S. Begin with a small number of samples (e.g., 10 points) to estimate $\hat{p}$. Then incrementally increase the samples and update $\hat{p}$ until the error bound γ is just acceptable for the selected confidence level.

2) Reduce the curvatures such that the increased probability Δp_s is a fraction (e.g., 10%) of the initial probability p_s. Sample the g function only in the incremental region ΔS.

3) Update the probability estimate using Eq. (10). Stop the analysis if the probability estimate converges based on Eq. (11). Otherwise, go back to step 2.

4) As an option (in case the MPP is in error), slightly reduce the minimum distance (i.e., shift the parabolic surface toward the origin) and generate additional samples in ΔS. If the MPP is correct, ΔS will be in the safe region. Stop the analysis if no more failure points are observed in ΔS.

AIS for System Reliability

In this paper, a system reliability problem is defined using a fault tree to provide a convenient and systematic way to manage multiple failure modes. Each bottom event is defined by an explicitly or implicitly defined function. Sequential failures can be modeled using a sequence of conditional limit-state functions under a PRIORITY AND gate structure. A sequence of g functions, corresponding to a sequence of updated structural configurations with load redistribution, can be explicitly or implicitly defined in the bottom events.

A failure mode can involve one or more limit states. By combining all of the failure modes, the system limit-state surface can be defined. In other words, the system limit-state surface can be constructed piece by piece. The AIS procedure for system reliability analysis requires the construction of multiple parabolic surfaces. In principle, it is a straightforward extension of the concept for one limit state. However, the challenge is to develop an optimal procedure to perturb limit states and generate incremental sampling regions.

The currently implemented approach adds samples progressively starting from the most important limit state (i.e., with maximum p_f), based on the AMV+ models. This approach is relatively simple but is not effective when the system failure is governed by the joint effects from several limit states and none of the limit states are dominant. In such cases, the MPP of the individual limit state is not a likely event for a system failure. A potentially more effective but more complicated procedure adds samples progressively based on the failure modes. The initial sampling regions are based on the approximate bottom events (limit states), and the increased sampling regions are based on the jointly perturbed bottom events.

Figure 2 illustrates the concept of the aforementioned more effective AIS procedure for a system with two failure modes that involve three limit states. The probability of failure statement is

$$p_f = P\{(g_1 < 0) \cup [(g_2 < 0) \cap (g_3 < 0)]\} \tag{12}$$

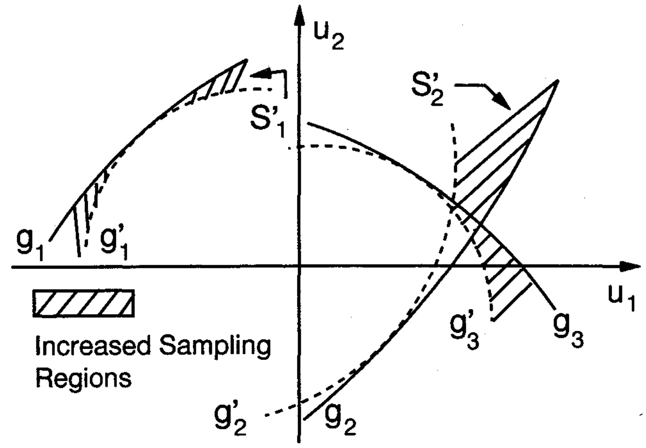

Fig. 2 Illustration of the AIS procedure for system reliability.

AIS-Based Reliability Sensitivity Analysis

Sensitivity with Respect to a Distribution Parameter

When a distribution parameter is changed, the sensitivity of p_f, abbreviated as p, with respect to a distribution parameter θ can be evaluated from

$$\frac{\partial p}{\partial \theta} = \int_\Omega \cdots \int \frac{\partial f_x}{\partial \theta}\, \mathrm{d}x \tag{13}$$

Therefore,

$$\frac{\partial p/p}{\partial \theta/\theta} = \int_\Omega \cdots \int \frac{\theta}{p} \frac{\partial f_x}{f_x \partial \theta} f_x \, \mathrm{d}x = E\left[\frac{\theta \partial f_x}{f_x \partial \theta}\right]_\Omega \tag{14}$$

or

$$\frac{\partial \ell n\, p}{\partial \ell n\, \theta} = E\left[\frac{\partial \ell n\, f_x}{\partial \ell n\, \theta}\right]_\Omega \tag{15}$$

where the subscript Ω denotes that the expected value is evaluated using the joint PDF in the failure region. In general, numerical differentiation methods can be used to compute the value within E[.]. The probabilistic sensitivities can be computed using those AIS points in the failure region. No additional g-function calculations are required.

Based on Eq. (15), two types of probabilistic sensitivity coefficients are proposed that are particularly useful for probabilistic design. One is the sigma (standard deviation) sensitivity coefficient $S_{\sigma i}$, and the other is the mean sensitivity coefficient $S_{\mu i}$. The two sensitivity coefficients are defined as

$$S_{\sigma i} = \frac{\partial p_i/p}{\partial \sigma_1/\sigma_i} \tag{16}$$

$$S_{\mu i} = \frac{\partial p_i/p}{\partial \mu_i/\sigma_i} \tag{17}$$

where μ_i and σ_i are the mean and the standard deviation, respectively, of the random variable X_i. In Eq. (17), the use of the standard deviation as a scale factor implies that the allowable design range of a mean value is limited to a local region characterized by the random variable variability. When one or more σ_i are very small (e.g., approaching zero) relative to their allowable μ_i design ranges, it may be more appropriate to replace σ_i by the allowable design ranges in Eq. (17).

Let the reliability be R, which equals $1 - p$. It can be easily shown that

$$\frac{\partial R_i/R}{\partial \theta_i/\theta} = -\frac{p}{R} S_{\theta_i} \tag{18}$$

which indicates that the sensitivity coefficients based on R are negatively proportional to the sensitivity coefficients based on p. In this paper, the sensitivity coefficients defined in Eqs. (16) and (17) will be called reliability sensitivity coefficients because they relate to reliability closely.

For some random variables, Eq. (15) can be simplified. For example, for independent normal variables, the reliability sensitivity coefficients become

$$S_{\sigma i} = E\left[\left(\frac{X_i - \mu_i}{\sigma_i}\right)^2 - 1\right]_\Omega = E[u_i^2 - 1]_\Omega \tag{19}$$

$$S_{\mu i} = E\left[\left(\frac{X_i - \mu_i}{\sigma_i}\right)\right]_\Omega = E[u_i]_\Omega \tag{20}$$

The proposed coefficients are dimensionless and can take positive, negative, or zero values. If desired, they can be normalized such that the sum of the normalized coefficients becomes one.

When an $S_{\sigma i}$ is zero or relatively small, it implies that the realization of X_i can be varied over a wide range without significantly changing p. This, in turn, implies that $S_{\mu i}$ will be negligible. On the other hand, when an $S_{\sigma i}$ is relatively large, the corresponding $S_{\mu i}$ will also tend to be significant. These suggest that $S_{\sigma i}$ and $S_{\mu i}$ are strongly related and both can be used to identify key contributing random variables. However, when the allowable μ_i design ranges are used to replace σ_i in Eq. (17), the resulting $S_{\mu i}$ may not be related to $S_{\sigma i}$.

For a single limit state, the relationship between the sensitivity factors α_i and $S_{\mu i}$ of a normal random variable can be approximated for large u_i^*, using the following relation:

$$S_{\mu i} = E[u_i]_\Omega \approx u_i^* \tag{21}$$

The approximation holds because the sampling PDF is concentrated near the MPP for large u_i^*. This leads to

$$S_{\mu i} \approx \beta \alpha_i \tag{22}$$

Therefore, for large u_i^*, S_{μ_i} is approximately proportional to α_i.

The two coefficients are useful for reliability-based design optimization. Assume that a small change in θ, denoted as $\Delta\theta$, results in a small change in p, Δp. From Eqs. (16) and (17), the changes can be approximated by

$$\Delta p_i \approx p S_{\sigma_i} \frac{\Delta\sigma_i}{\sigma_i} \tag{23}$$

$$\Delta p_i \approx p S_{\mu_i} \frac{\Delta\mu_i}{\sigma_i} \tag{24}$$

A random variable approaches a deterministic value if σ_i is reduced to 0, i.e., $\Delta\sigma_i = -\sigma_i$. In this case,

$$\Delta p_i \approx -p S_{\sigma_i} \tag{25}$$

Assuming the approximation holds even when all of the standard deviations are reduced to zero, the probability of failure should be zero, and $\Sigma\Delta p_i = -p$, which results in

$$\sum_{i=1}^{n} S_{\sigma_i} \approx 1 \tag{26}$$

Although the preceding conclusion is based on a hypothetical assumption, Eq. (26) suggests that $S_{\sigma i}$ could be used to rank the importance of the random variable uncertainty (σ_i). In practice, if any S_{σ_i} is significantly greater than one, then it is an indication that the relationship between p and σ_i is strongly nonlinear and the use of Eqs. (23) and (24) must be limited to small changes.

From Eqs. (23) and (24), p can be decreased by either changing the variabilities or by shifting the mean values. Reliability improvements could be achieved by a combination of various design modifications such as tightening tolerances, using better materials, or changing geometries. A decision analysis based on the cost, manufacturing, and other design considerations may be performed to select an optimal design. Upon a design change, a reanalysis should generally be conducted to confirm the improved reliability.

The usefulness of the AIS-based sensitivity analysis approach is fully realized for system reliability problems in which there are multiple limit states and therefore multiple MPPs. The sensitivity calculations using the AIS approach are straightforward and more accurate relative to analytically combining the sensitivities from the multiple MPPs.

A special case that is useful in practical applications is when each failure mode is represented by a limit state and the failure modes are independent or weakly correlated. In this case, it may be sufficient to apply the AIS method to each individual failure mode and use the results for system reliability and reliability sensitivity analysis by using the probability bounds. For example, from the probability theory, the unimodal upper bound is

$$p \le p_u = \sum_{j=1}^{m} p_j \tag{27}$$

where p_u is the upper bound and p_j is the probability of failure for the jth failure mode. The bimodal bounds,[6] which are narrower than the unimodal bounds, may not be practical to use because of the requirements in computing joint probabilities between every two failure modes.

When p_j ($j = 1, m$) are small and the failure modes are independent or weakly correlated, the second- and higher-order joint probabilities between the failure modes will be insignificant relative to p_j ($j = 1, m$) and therefore $p \approx p_u$. By reducing a σ_i to zero, the total probability change is

$$\Delta p \approx \sum_{j=1}^{m} p_j S_{\sigma_i}^j \frac{\Delta\sigma_i}{\sigma_i} \approx p S_{\sigma_i} \frac{\Delta\sigma_i}{\sigma_i} \tag{28}$$

where $S_{\sigma_i}^j$ is the sigma sensitivity coefficient for the jth failure mode and S_{σ_i} is the sigma sensitivity coefficient for the system. Therefore,

$$S_{\sigma_i} \approx \sum_{j=1}^{m} \frac{p_j}{p} S_{\sigma_i}^j \tag{29}$$

In other words, p_j/p are the weighting factors in computing the system sensitivity coefficients.

Similarly,

$$S_{\mu_i} \approx \sum_{j=1}^{m} \frac{p_j}{p} S_{\mu_i}^j \tag{30}$$

When the failure modes are significantly correlated, the upper bound may not be sufficiently accurate. In such cases, it is more accurate to apply the AIS method to the combined system failure regions to compute system reliability sensitivities.

Sensitivity with Respect to a g-Function Parameter

In reliability-based design, some design parameters can be adjusted. These parameters might be deterministic, such as tightly controlled geometries. To compute reliability sensitivity with respect to such a parameter, one could build on the previous AIS samples to compute Δp due to a limit-state surface perturbation with perhaps some extra samples for convergence. An alternative, designed to provide estimates without extra samples, is proposed as follows.

To use the results developed earlier, one could add an artificial variability (i.e., standard deviation). The assumed σ_i should be small enough not to produce a dominant $S_{\sigma i}$. On the other hand, the assumed σ_i should be large enough so that $S_{\mu i}$ could be estimated with a reasonable number of AIS samples.

Based on the assumed σ_i, the correct probability can be recovered by using Eq. (25), assuming that Δp is reasonably small with respect to p. Further work is in progress to determine criteria for selecting a proper standard deviation.

Numerical Example

A numerical example is presented to demonstrate the AIS-based sensitivity analysis method. The structure is a turbine blade subject to high temperatures and pressures. Three failure modes are considered. The fault tree representation of the structural failure is given in Fig. 3.

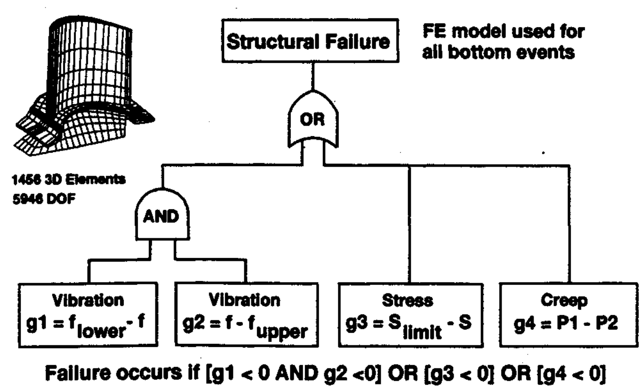

Fig. 3 Fault tree model for the turbine blade reliability analysis.

Failure of the structure is defined as follows:

1) The first natural frequency falls within an operating range of f_{lower} to f_{upper}, causing an intolerably large vibration response. The two limit-state functions are

$$g_1 = f_{\text{lower}} - f \quad \text{and} \quad g_2 = f - f_{\text{upper}} \tag{31}$$

2) The von Mises stress exceeds a limit. The limit-state function is

$$g_3 = \sigma_{\text{limit}} - \sigma \tag{32}$$

where σ is the von Mises stress at a highly stressed location. In this analysis $\sigma_{\text{limit}} = 110$ ksi.

3) Creep at a specified time to rupture exceeds the Larson-Miller parameter. The limit-state function is

$$g_4 = B_0 + B_1 \log \sigma + B_2 \log \sigma^2 + B_3 \log \sigma^3 - T(C + \log t) \tag{33}$$

where B_0, B_1, B_2, B_3, and C are material coefficients, T is the absolute temperature, and t is the service time. In this analysis, $t = 75$ h.

The blade has directional material properties. The random variables include three material orientation angles, elastic modulus, Poisson's ratio, shear modulus, a creep material coefficient (B_0), and the density. The random variable definitions and the failure modes to which they contribute are given in Table 1.

The finite element model consists of 1456 three-dimensional elements. The blade is subjected to centrifugal, pressure, and temperature loadings. The material properties are temperature dependent.

The system reliability analyses were carried out using NESSUS.[19,20] The AIS analyses were based on 90% confidence interval and 10% error bound. Two cases were compared. Both cases required the development of the AMV+ models for all of the limit states using the finite element models. Afterward, the first case applied AIS to the AMV+ models and the second results applied AIS to the finite element models. The results shown in Table 2 indicate that the AMV+ models produced reasonably accurate results with significantly less computer time.

The system reliability analyses indicated that the g_2 limit state has no significant effect on the reliability. To simplify the reliability sensitivity analysis, g_2 was removed, leaving only three limit states in the problem. For demonstration purposes, the AMV+ models (i.e., case 1 in Table 2) were used to validate the AIS-based sensitivity analysis methodology. Table 3, taken from Ref. 19, lists the coefficients [see Eq. (5)] of the three AMV+ models used in generating the AIS samples and computing the reliability sensitivities. It should be noted that, in general, finite element models (e.g., case 2 in Table 2) should be used if the accuracies of the AMV+ models are questionable.

Based on the AMV+ models defined by the data in Table 3, 85 AIS samples were generated in which 73 samples were in the failure region. These failure points were used to compute the reliability sensitivity coefficients for the system and individual failure modes; results are shown in Table 4.

The 73 failure points were split into three failure modes. There were three common points between the stress failure mode and the vibration failure mode, indicating a small correlation with approximately a joint probability of 0.00074.

Table 1 Random variable definitions

Variable	Mean	Std. dev	Distribution	Failure mode
Material orientation θ_z, deg	0.05236	0.067544	Normal	All
Material orientation θ_y, deg	−0.03491	0.067544	Normal	All
Material orientation θ_x, deg	0.08727	0.067544	Normal	All
Elastic modulus E[a]	18.36E6	0.4595E6	Normal	All
Poisson's ratio ν[a]	0.386	0.00965	Normal	All
Shear modulus G[a]	18.63E6	0.4657E6	Normal	All
Creep coef. B_0	86.0	0.086	Normal	Creep
Density ρ	0.805E-3	0.493E-5	Normal	Vibration

[a]At room temperature.

Table 2 AIS system reliability results (< 10% error, 90% confidence)

Case	Structural models	Number of AIS system samples	Probability of failure	Total CPU[a], h
1	Approx. (AMV+)	85 (73 failures)	0.018	3.7
2	Finite element	117 (85 failures)	0.022	25

[a]Including the AMV+ solution time using the finite element model. CPU time based on an HP 720 workstation.

Table 3 Limit state definitions at the MPPs

		θ_Z	θ_Y	θ_X	E	ν	G	B_0	ρ
Coefficients	b_0	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
Frequency	0.140E4	−0.287E3	−0.304E3	0.525E2	−0.103E-3	−0.233E4	−0.358E-4	0.0	0.276E7
Stress	0.151E6	−0.368E5	−0.437E5	0.152E3	−0.124E-2	−0.207E6	−0.187E-2	0.0	0.0
Creep	−0.781E5	0.997E3	0.389E4	−0.591E3	−0.245E-3	−0.119E4	−0.113E-3	0.100E4	0.0

Table 4 AIS-based reliability sensitivity coefficients (numbers in bold indicate the top three coefficients in magnitude)

Failure mode	p	β	No. of samples	Coef.	θ_z	θ_y	θ_x	E	ν	G	B_0	ρ
Vibration	0.00528	2.556	22[b]	u^*	0.78	**0.83**	−0.144	**1.91**	**0.90**	0.67	—	−0.55
				S_μ	0.64	**1.08**	−0.16	**1.96**	**1.45**	0.71	—	−0.83
				S_σ	0.18	**0.59**	0.48	**3.14**	**2.09**	0.45	—	0.53
Stress	0.00233	2.829	12[b]	u^*	**1.57**	**1.87**	−0.01	0.36	**1.27**	0.55	—	—
				S_μ	**1.58**	**2.09**	0.24	0.36	**1.35**	0.99	—	—
				S_σ	**1.85**	**4.16**	−0.52	0.23	**1.76**	1.07	—	—
Creep	0.00978	2.334	42	u^*	−0.50	**−1.95**	0.29	**0.84**	0.08	−0.39	**−0.64**	—
				S_μ	−0.57	**−2.33**	0.41	**0.86**	0.19	0.41	**−0.65**	—
				S_σ	0.43	**4.63**	0.01	**0.57**	0.34	0.19	**0.46**	—
System	0.0181	2.095[a]	73[c]	S_μ	0.44	**−0.72**	0.23	**1.08**	**0.65**	0.58	−0.35	−0.37
				S_σ	0.49	**3.48**	0.07	**1.24**	**0.91**	0.44	0.19	0.16

[a]Calculated from $\beta = \Phi^{-1}(p)$. [b]Three common points in the frequency and stress failure modes. [c]Total number of AIS samples is 85.

Table 5 Probability prediction using sensitivity coefficients

Changes, %		Probabality of failure p_f Predicted	Exact
$\Delta\sigma_{\theta_y}/\sigma_{\theta_y}$	−5	0.0149	0.0152
	−10	0.0118	0.0127
	−20	0.0055	0.0090
$\Delta\mu_E/\sigma_E$	−5	0.0171	0.0161
	−10	0.0161	0.0153
	−20	0.0142	0.0138

The first three significant random variables are highlighted in Table 4. The results support the earlier approximation about $S_{\mu_i} \approx u_i^*$ [Eq. (21)] for large u_i^*.

Because the correlations between the failure modes are small, we can verify Eq. (29). For the θ_y variable, the predicted S_σ for the system is

$$S_\sigma = \frac{0.00528}{0.0181}(0.59) + \frac{0.00233}{0.0181}(4.16) + \frac{0.00978}{0.0181}(4.63) = 3.21 \tag{34}$$

which is close to the reference value of 3.48 in Table 4.

The sensitivity coefficients can be used to predict probability changes due to small changes in μ_i or σ_i. The results of several test cases based on the most significant variables, θ_y and E, are listed in Table 5. Fairly good agreement between the predicted and the exact solutions was observed for up to 10% changes in the standard deviations.

Summary

Recent developments in efficient structural reliability analysis methods were presented. The advanced mean-value-based method and the adaptive importance sampling method allow the analysis to focus on critical failure regions. Using an example of a turbine blade reliability analysis, it has been demonstrated that the AIS method can be used to compute system reliability and reliability sensitivities efficiently and accurately. Two reliability sensitivity coefficients were proposed that can be easily computed using the AIS-based sampling points. The AIS-based method can be used to support reliability/risk-based design and decisions and is particularly suitable for analyzing computer-intensive models.

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References

[1]Wu, Y.-T., Millwater, H. R., and Cruse, T. A., "An Advanced Probabilistic Structural Analysis Method for Implicit Performance Functions," *AIAA Journal*, Vol. 28, No. 9, 1990, pp. 1663–1669.

[2]Rajagopal, K. R., Keremes, J., Ho, H., and Orient, G., "A Probabilistic Approach to the Evaluation of Fatigue Damage in a Space Propulsion System Injector Element," *Proceedings of the AIAA/ASME/ASCE/AHS/ASC 33rd Structures, Structural Dynamics, and Materials Conference*, AIAA, Washington, DC, 1992, pp. 625–639.

[3]Torng, T. Y., Wu, Y.-T., and Millwater, H.R., "Structural System Reliability Calculation Using a Probabilistic Fault Tree Analysis Method," *Proceedings of the AIAA/ASME/ASCE/AHS/ASC 33rd Structures, Structural Dynamics, and Materials Conference*, AIAA, Washington, DC, 1992, pp. 603–613.

[4]Wu, Y.-T., "An Adaptive Importance Sampling Method for Structural System Reliability Analysis," *Reliability Technology, Proceedings of the Symposium on Reliability Technology*, edited by T. A. Cruse, AD-Vol. 28, American Society of Mechanical Engineers, New York, 1992, pp. 217–231.

[5]Madsen, H. O., Krenk, S., and Lind, N. C., *Methods of Structural Safety*, Prentice-Hall, Englewood Cliffs, NJ, 1986.

[6]Ang, A. H.-S., and Tang, W. H., *Probability Concepts in Engineering Planning and Design, Volume II: Decision, Risk, and Reliability*, Wiley, New York, 1984.

[7]Wu, Y.-T., Burnside, O. H., and Cruse, T. A., "Probabilistic Methods for Structural Response Analysis," *Computational Mechanics of Reliability Analysis*, edited by W. K. Liu and T. Belytschko, Elmepress International, VA, 1989, Chap. 7.

[8]Wu, Y.-T., Gureghian, A. B., Codell, R. B., and Sagar, B., "Sensitivity and Uncertainty Analysis Applied to One-Dimensional Transport in a Layered Fractured Rock, Part II: Probabilistic Methods Based on the Limit-State Approach," *Nuclear Technology*, Vol. 104, No. 2, 1993, pp. 297–308.

[9]Harbitz, A., "An Efficient Sampling Method for Probability of Failure Calculation," *Structural Safety*, Vol. 3, Jan. 1986, pp. 109–115.

[10]Bjerager, P., "Probability Integration by Directional Simulation," *Journal of Engineering Mechanics*, Vol. 114, No. 8, 1987, pp. 1285–1302.

[11]Ditlevsen, O., Bjerager, P., Olesen, R., and Hasofer, A., "Directional Simulation in Gaussian Processes," *Probabilistic Engineering Mechanics*, Vol. 3, No. 4, 1988, pp. 207–217.

[12]Melchers, R. E., "Importance Sampling in Structural Systems," *Structural Safety*, Vol. 6, No. 1, 1989, pp. 3–10.

[13]Melchers, R. E., "Radial Importance Sampling for Structural Reliability," *Journal of Engineering Mechanics*, Vol. 116, No. 1, Paper 24288, 1990, pp. 189–203.

[14]Engelund, S., and Rackwitz, R., "Comparison of Some Importance Sampling Techniques in Structural Reliability," *Proceedings of the 1992 ASCE Specialty Conference on Probabilistic Mechanics*, American Society of Civil Engineers, New York, 1992, pp. 108–111.

[15]Wu, Y.-T., and Burnside, O. H., "Computational Methods for Probability of Instability Calculations," *Proceedings of the AIAA/ASME/ASCE/AHS/ASC 31st Structures, Structural Dynamics, and Materials Conference*, AIAA, Washington, DC, 1990, pp. 1081–1091.

[16]Wu, Y.-T., Torng, T. Y., Millwater, H. R., Fossum, A.F., and Rheinfurth, M. H., "Probabilistic Methods for Rotordynamics Analysis," SAE Transactions—Journal of Aerospace, Sec. 1, Vol. 100, Pt. 2, 1991, Society of Automotive Engineers, Inc. 1991, pp. 261–272.

[17]Der Kiureghian, A., Lin, H.-Z., and Hwang, S. J., "Second-Order Reliability Approximations," *Journal of Engineering Mechanics*, Vol. 113, No. 8, Paper 21727, 1987, pp. 1208–1225.

[18]Tvedt, L., "Distribution of Quadratic Forms in Normal Space— Application to Structural Reliability," *Journal of Engineering Mechanics*, Vol. 116, No. 6, 1990, pp. 1183–1197.

[19]Millwater, H. R., Wu, Y.-T., Torng, Y., Thacker, B., Riha, D., and Leung, C., "Recent Developments of the NESSUS Probabilistic Structural Analysis Computer Program," *Proceedings of the AIAA/ASME/ASCE/AHS/ASC 33rd Structures, Structural Dynamics, and Materials Conference*, AIAA, Washington, DC, 1992, pp. 614–624.

[20]Millwater, H. R., and Wu, Y.-T., "Computational Structural Reliability Analysis of a Turbine Blade," 38th ASME International Gas Turbine and Aeroengine Congress and Exposition, 1993 (ASME Paper 93-GT-237).